DGPS TSPI PROJECT
Coordinate Conversion from Geodetic to Measurement Frame

OBJECTIVE

The objective is to present equations which convert geodetic position data (i.e., latitude, longitude, and altitude) derived from a GPS receiver on board an aircraft to rectangular coordinates associated with a measurement program. Both exact and approximate equations are presented.

MATHEMATICAL PRELIMINARIES

Earth Datum Model

The WGS-84 model for the earth is an oblate ellipsoid with semi-major axis *a* given by

$$a = 6{,}378{,}137.0 \ m \qquad \textbf{Eq. 1}$$

Distance *a* is converted to English units using 39.37 in./m, the appropriate conversion factor for geodetic survey applications. The result is

$$a = 20{,}925{,}604.5 \ ft \ (US \ Survey) \qquad \textbf{Eq. 2}$$

The semi-minor axis *b* is not defined. Instead the flattening *f* is specified as

$$f = \frac{a-b}{a} = \frac{1}{298.257{,}223{,}563} \qquad \textbf{Eq. 3}$$

Earth Parameters Used for Calculations

For computational formulas, the eccentricity *e* (actually the square of the eccentricity e^2) is used rather than the flattening,

$$e^2 = 1 - \left(\frac{b}{a}\right)^2 = 2f - f^2 = 0.00669437999014131 7 \qquad \textbf{Eq. 4}$$

Instead of the semi-major axis, computational formulas employ the "radius of curvature in the prime vertical" $r_p = r_p(L)$

$$r_p = r_p(L) = \frac{a}{\sqrt{1 - e^2 \sin^2(L)}} \qquad \textbf{Eq. 5}$$

Here L is the latitude of the location involved. Note that r_p is not constant, but instead varies with location (e.g., the radius of curvature is different at the aircraft and reference point locations); the notation $r_p(L)$ is intended to reinforce the fact that r_p is a function of L.

Reference Points and Aircraft Geodetic Positions

Two reference points (or waypoints) are used. The first (and main) reference point is the origin for measurement program coordinate system. It has latitude L_0, longitude 8_0, and altitude h_0, all relative to the WGS-84 ellipsoid. The second reference point is used to define the azimuth direction of the measurement program local coordinate frame. It has latitude L_1, longitude 8_1, and altitude h_1.

The aircraft has latitude L_a, longitude 8_a, and altitude h_a.

EXACT EQUATIONS

Positions in Earth-Centered Earth-Fixed (ECEF) Frame

The earth-centered earth-fixed (ECEF) frame, with axes x, y and z, is defined as follows:

- x lies in plane of the equator, positive sense points 90 deg east of the Greenwich meridian
- y coincides with spin axis, positive sense points toward north pole
- z lies in plane of the equator, positive sense points toward the Greenwich meridian

Vectors coordinatized in the ECEF frame have a superscript e.

The position of first reference point (waypoint), expressed in the ECEF frame, is

$$\mathbf{p}_0^e = \begin{bmatrix} p_{0x} \\ p_{0y} \\ p_{0z} \end{bmatrix} = \begin{bmatrix} [r_p(L_0) + h_0] \cos L_0 \sin\lambda_0 \\ [(1-e^2) r_p(L_0) + h_0] \sin L_0 \\ [r_p(L_0) + h_0] \cos L_0 \cos\lambda_0 \end{bmatrix} \qquad \textbf{Eq. 6}$$

The aircraft position expressed in the ECEF frame is

$$\mathbf{p}_a^e = \begin{bmatrix} p_{ax} \\ p_{ay} \\ p_{az} \end{bmatrix} = \begin{bmatrix} [r_p(L_a) + h_a] \cos L_a \sin\lambda_a \\ [(1-e^2) r_p(L_a) + h_a] \sin L_a \\ [r_p(L_a) + h_a] \cos L_a \cos\lambda_a \end{bmatrix} \qquad \textbf{Eq. 7}$$

Observe that the two radii of curvature, $r_p(L_0)$ and $r_p(L_a)$, are, in general, not equal because they correspond to different latitudes. The difference vector

$$\Delta\mathbf{p}^e = \mathbf{p}_a^e - \mathbf{p}_0^e = \begin{bmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \end{bmatrix} = \begin{bmatrix} p_{ax} - p_{0x} \\ p_{ay} - p_{0y} \\ p_{az} - p_{0z} \end{bmatrix} \qquad \textbf{Eq. 8}$$

is the position of the aircraft relative to the first reference point in the ECEF frame.

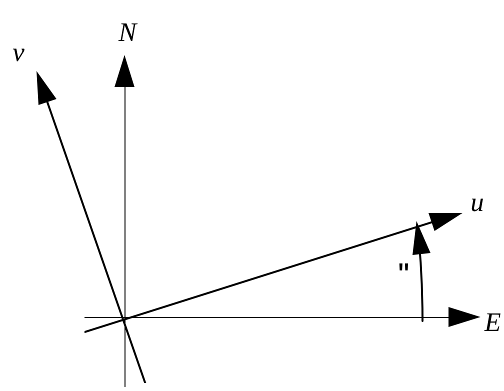
North-Pointing Frame

The north-pointing frame is level (that is, has two axes which are tangent to the WGS-84 ellipsoid) at the latitude and longitude of the first reference point. Its coordinate axes are: East, E; North, N; and Up, U.

The position of the aircraft relative to the first reference point, coordinatized in the north-pointing frame is

$$\begin{bmatrix} \Delta p_E \\ \Delta p_N \\ \Delta p_U \end{bmatrix} = \begin{bmatrix} \cos\lambda_0 & 0 & -\sin\lambda_0 \\ -\sin L_0 \sin\lambda_0 & \cos L_0 & -\sin L_0 \cos\lambda_0 \\ \cos L_0 \sin\lambda_0 & \sin L_0 & \cos L_0 \cos\lambda_0 \end{bmatrix} \begin{bmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \end{bmatrix} \qquad \textbf{Eq. 9}$$

Observe that the 3x3 matrix in Eq. 9 is constant for a given test program; thus these trigonometric functions need not be computed in real time.



Measurement Program Frame

The measurement program frame has its origin at the latitude and longitude of the first reference point. Its coordinate axes are u, v and w. Axis u is level, and its positive sense points toward the second reference point. It is expected that, in most applications, u will be the along-runway axis. Angle α is subtended by axes E and u and is positive when u is north of E (see figure). Axis v is also level; it is orthogonal to u with positive sense $\pi/2$ radians counterclockwise from u. In most applications, v will be the cross-runway axis. Axis w is vertical (same as U).

When angle α is known, the position of the aircraft relative to the first reference point, coordinatized in the measurement program frame, is given by

$$\begin{bmatrix} \Delta p_u \\ \Delta p_v \\ \Delta p_w \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta p_E \\ \Delta p_N \\ \Delta p_U \end{bmatrix} = \begin{bmatrix} \Delta p_E \cos\alpha + \Delta p_N \sin\alpha \\ -\Delta p_E \sin\alpha + \Delta p_N \cos\alpha \\ \Delta p_U \end{bmatrix} \qquad \textbf{Eq. 10}$$

To find angle α, use Eqs. 6-9, with the exception that, in Eq. 7, the latitude L_1 and longitude λ_1 of the second reference point are used in place of those for the aircraft. Denote the result from Eq. 9 by

$$\begin{bmatrix} \Delta p_E' \\ \Delta p_N' \\ \Delta p_U' \end{bmatrix} \qquad \textbf{Eq. 11}$$

Then α is given by

$$\alpha = \arctan(\Delta p_N', \Delta p_E') \qquad \textbf{Eq. 12}$$

where the two-argument arctan function has arguments (opposite_side, adjacent_side).

Note that α is constant for a given test program. Thus, in Eq. 10, the sin/cos terms are also constant and do not have to be computed in real time.

APPROXIMATE EQUATIONS

The equations below were derived from the exact equations by expanding all trigonometric (sin/cos) expressions in Taylor series about the first reference point. As a result, the trigonometric functions in Eq. 7 do not have to be computed in real-time; only addition and multiplication are done in real time to implement this technique. Numerical testing has revealed that in the Taylor series expansion, terms up to and including second-order must be retained in order to reduce the approximation error to less than 1 ft for an aircraft within 15 mi of the first reference point.

In place of Eqs. 6-9, the expression for each coordinate in the north-pointing frame is computed from a sum of products, where each product is a constant (A, B, etc.) times the difference between the aircraft and first reference point latitude, longitude, and/or altitude. Latitudes and longitudes which are "free standing" (i.e., not the argument of a trigonometric function) must be expressed in radians.

$$\begin{aligned}\Delta p_E &= A_E(\lambda_a - \lambda_0) + B_E h_a(\lambda_a - \lambda_0) - C_E(L_a - L_0)(\lambda_a - \lambda_0)\\ A_E &= r_p(L_0)\cos(L_0)\\ B_E &= \cos(L_0)\\ C_E &= r_p(L_0)\sin(L_0)\end{aligned} \qquad \textbf{Eq. 13}$$

$$\begin{aligned}\Delta p_N &= A_N(L_a - L_0) + h_a(L_a - L_0) + B_N(\lambda_a - \lambda_0)^2\\ A_N &= r_p(L_0)\,[1 - e^2\cos^2(L_0)]\\ B_N &= \frac{1}{4}r_p(L_0)\sin(2L_0)\end{aligned} \qquad \textbf{Eq. 14}$$

$$\begin{aligned}\Delta p_U &= (h_a - h_0) - A_U(L_a - L_0)^2 - B_U(\lambda_a - \lambda_0)^2\\ A_U &= \frac{1}{2}r_p(L_0)\\ B_U &= \frac{1}{2}r_p(L_0)\cos^2(L_0)\end{aligned} \qquad \textbf{Eq. 15}$$

As is the case for the exact equations, $r_p(L_0)$ denotes the radius of curvature for the first waypoint. Also as for the exact method, Eq. 10 is used to find the aircraft position in the measurement frame from its East-North-Up coordinates.